INVESTIGATIONES LINGUISTICAE, VOL. XIV; POZNAŃ, DECEMBER 2006

Zermelo: a Well Founded Antiskolemism*

Jerzy Pogonowski†

Institute of Linguistics, Adam Mickiewicz University
ul. Międzychodzka 5, 60-371 Poznań, POLAND

pogon@amu.edu.pl

In his response to the letter from Paul Bernays (with congratulations on the occasion of his 70th anniversary) Zermelo wrote (October 1, 1941):

> Man wird eben immer einsamer, ist aber umso dankbarer für jedes freundliche Gedanken. [...] Wo mein Name noch genannt wird, geschieht es immer *nur* in Verbindung mit dem 'Auswahlprinzip', auf das ich *niemals* Prioritätsanspruche gestellt habe. [...] Dabei erinnere ich mich, daß schon bei der Mathematiker-Tagung in Bad Elster mein Vortrag über Satz-Systeme durch eine Intrige der von Hahn und Gödel vertretenen Wiener Schule von der Diskussion ausgeschlossen wurde, und habe seitdem die Lust verloren, über Grundlagen vorzutragen. So geht es augenscheinlich jedem, der keine 'Schule' oder Klique hinter sich hat. Aber vielleicht kommt noch eine Zeit, wo auch meine Arbeiten wieder entdeckt und gelesen werden.[1]

Zermelo had in mind here his works listed below. They concern a foundational program formulated by him with special emphasis put on the *infinitary* (though always well founded) nature of mathematical proof. This idea is of course in sharp opposition to the (quite well established at that time) common uderstanding of the notion of finitary formal proof. Zermelo rejects what he himself calls *Skolemism* and *the finitary prejudices*: the views that set theory should be axiomatized in a first order language (which implies that quantification over propositional functions in the comprehension axiom would be out of question) and that mathematical theories in general should be codified solely in terms of finitary logic.

In the formulation of his program Zermelo makes an essential use of hierarchies of well founded domains described in his paper *1930*. Besides the distinction between *closed* and *open* domains this paper brings categorical characterizations of models of set theory (with respect to two numerical parameters — the number of urelements and the ordinal rank of the domain). It contains also an important observation that the hierarchies V_κ, where κ is strongly inaccessible, form natural models for set theory (Zermelo works here in set theory with a second order axiom of comprehension, with

*Presented at the conference *Applications of Logic to Philosophy and the Foundations of Mathematics VIII*, Karpacz, April 2004. Institute of Linguistics, AMU, did not support my participation in this conference. I am indebted to Professor Jan Zygmunt for the invitation as well as for his kind financial support of participation of the staff of The Department of Applied Logic, Adam Mickiewicz University in this conference series.

†I warmly thank Alexander von Humboldt-Stiftung and *Fachbereich Philosophie der Universität Konstanz*, Germany, for providing me with excellent conditions for my research work in Konstanz during the Spring of 2003 — the present abstract is one of the results of that work. The work on this abstract was also sponsored by the research project KBN 2H01A 00725 *Metody nieskończonościowe w teorii definicji* (*Infinitary methods in the theory of definitions*) headed by Professor Janusz Czelakowski at the Institute of Mathematics and Information Science, University of Opole, Poland.

[1] Cf. Peckhaus 1990, p. 20.

a version of the replacement axiom, with the axiom of foundation, without the axiom of infinity and without the axiom of choice — the latter is assumed in the metatheory, as a logical principle).

Any mathematical theory is represented, according to Zermelo, by: an infinite *domain*, a collection of *fundamental relations* over that domain and a collection of *truth partitions* of the formulas of this theory (making its axioms true). Formulas may be treated as (well founded!) *sets*; infinite conjunctions and disjunctions are allowed. Zermelo's notion of *proof* corresponds, in a sense, to that what is now commonly understood by *logical consequence*. Incompleteness in his sense is different from that of Gödel. Zermelo believed in decidability of all mathematical problems; however, he was aware that there are systems in which some true *Zermelo's sentences* have *Zermelo's proofs* which lie beyond those systems themselves.

At the time when Zermelo originated his program, it had little, if any, chances to be fully developed. The standard of finitary first order logic was winning, due to the achievements of Skolem and Gödel. Two decades later the situation looked different: Tarski, Henkin and Karp started the investigation of infinitary languages and Mostowski introduced generalized quantifiers. In the sixties Lindström developed Mostowski's approach and in the early seventies Barwise proposed and elaborated a whole domain of research in infinitary logic, soft model theory and admissible structures. It seems that the systems of logic based on admissible set theory are the closest counterparts to the original ideas of Zermelo. *Also, doch ist die Zeit angekommen...*

References to Zermelo

1921 Thesen über das Unendliche in der Mathematik. *Nachlaß*, published in van Dalen and Ebbinghaus 2000.

1929 Neun Vorträge über die Grundlagen der Mathematik (Universität Warschau, 27. Mai – 8 Juni 1929). The first and fourth of these abstracts were published as an appendix in Moore 1980.

1929a Über den Begriff der Definitheit in der Axiomatik. *Fundamenta Mathematicae* **14**, 339–344.

1930 Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre. *Fundamenta Mathematicae* **16**, 29–47.

1930a Über die logische Form der mathematischen Theorien. *Annales de la société polonaise de mathématiques* **9**, 187.

1931(?) Bericht an die Notgemeinschaft der Deutschen Wissenschaft über meine Forschungen betreffend die *Grundlagen der Mathematik*. *Nachlaß*, published as an appendix to Moore 1980.

1931a Letters to Gödel: September, 21 (published in Dawson 1985) and October, 29 (published in Grattan-Guinness 1979).

1932 Über Stufen der Quantifikation und die Logik des Unendlichen. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **41**, 85–92.

1932a Über mathematische Systeme und die Logik des Unendlichen. *Forschungen und Fortschritte* **8**, 6–7.

1935 Grundlagen einer allgemeinen Theorie der mathematischen Satzsysteme (Erste Mitteilung). *Fundamenta Mathematicae* **25**, 135–146.

1937 Der Relativismus in der Mengenlehre und der sogenannte Skolem'sche Satz. *Nachlaß*, published in van Dalen and Ebbinghaus 2000.

Selected Other References

Barwise, J. 1975. *Admissible Sets and Structures. An Approach to Definability Theory*. Berlin Heidelberg New York: Springer Verlag.

Barwise, J., Feferman, S. (Eds.) 1985. *Model-Theoretic Logics*. New York Berlin Heidelberg Tokyo: Springer Verlag.

van Dalen, D., Ebbinghaus, H.D. 2000. Zermelo and the Skolem Paradox. *The Bulletin of Symbolic Logic* Volume **6**, Number **2**, 145–161. Contains [Zermelo 1921] and [Zermelo 1937].

Dawson, J.W. 1985. Completing the Gödel–Zermelo Correspondence. *Historia Mathematica* **12**, 66–70.

Grattan-Guinness, I. 1979. In memoriam Kurt Gödel: his 1931 correspondence with Zermelo on his incompletability theorem. *Historia Mathematica* **6**, 294–304.

Moore, G.H. 1980. Beyond First-order Logic: The Historical Interplay between Mathematical Logic and Axiomatic Set Theory. *History and Philosophy of Logic* **1**, 95–137. Contains [Zermelo 1931(?)] and fragments of [Zermelo 1929].

Moore, G.H. 2002. Die Kontroverse zwischen Gödel und Zermelo. In: B. Buldt u.a. (Hrsg.) *Kurt Gödel. Wahrheit und Beweisbarkeit.* Band 1: *Dokumente und historische Analysen*, Band 2: *Kompendium zum Werk*. Wien: öbv&hpt VerlagsgmbH & Co., 55–64.

Peckhaus, V. 1990. 'Ich habe mich wohl gehütet, alle Patronen auf einmal zu verschießen'. Ernst Zermelo in Göttingen. *History and Philosophy of Logic* **11**, 19–58.

Taylor, R.G. 2002. Zermelo's Cantorian theory of systems of infinitely long propositions. *The Bulletin of Symbolic Logic* Volume **8**, Number **4**, 478–515.

Uzquiano, G. 1999. Models of second-order Zermelo set theory. *The Bulletin of Symbolic Logic* Volume **5**, Number **3**, 289–302.