CATEGORY THEORY IN GEOGRAPHY?

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Abstract: Is mathematical category theory a unifying tool for geography? Here we look at a few basic category theoretical ideas and interpret them in geographic example. We also offer links to indicate how category theory has been used as such in other disciplines. Finally, we announce the direction of our research program on this topic as a way to facilitate the learning, and maintenance of learning, of GIS software – and in the spirit of *Quaestiones Geographicae*, invite debate, comment, and contribution to this program in spatial mathematics.

Key words: category theory, mathematics, geography, Geographic Information Systems, web mapping, commutative diagrams

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Introduction

Category theory is a branch of mathematics with roots in the twentieth century and development from the mid–20th century on (Eilenberg, Mac Lane 1945). It is a language that visualizes the unity of mathematics as it economizes on thought and expression. Thus, it may expose previous unseen connections among disparate branches of mathematics. Because it is a general language, with an embedded principle of duality, a result that is proved in category theory may generate numerous results in a variety of subordinate mathematical areas. In fact, it can happen that when categories are mapped to each other, seemingly difficult or intractable problems yield unseen solutions.

Mark and Smith (2003) noted that categories (as a general word) are essential in human cognition and in the development of geographic information science and spatial data transfer. Qi et al. (2006) have offered fuzzy sets and related categorizations as a way to consider soil science classes. Works such as these share common elements in looking for clear ways to communicate materials involving large data sets. We appreciate that concern and concur that it is an important and exciting frontier. Here, we seek to employ a different meaning for ″category″ and to cast it in the broader spatial mathematics context (Arlinghaus, Kerski 2013).

The word ″category″ is yet another in the set of common words that may have uncommon meanings. We choose here to adopt the mathematical meaning of category, as in category theory developed by Samuel Eilenberg and Saunders Mac Lane in the 1940s (Eilenberg, Mac Lane 1945; Mac Lane 1971). A mathematical category is a broad, abstract structure that deals with mathematical structures and the relationships connecting them – as arrows and objects. Categories can offer a unifying framework of a high level of abstraction in which to place large amounts of information about both the objects and the arrows/relationships. The most important property of the arrows is that they can be ″composed″, that is, arranged in a sequence to form a new arrow. Arrows dominate: here, as in life, connections matter.

Definition

A category (illustrated in Fig. 1) is composed of:
- a class of objects,
- a class of morphisms, or ″arrows″; each arrow, *f*, has one object as the source object of *f* and one object as the target object of *f*,
- a binary operation, ∘ , called composition of arrows, such that for any three objects, *A*, *B*, and *C*, the composition of $f{:}A\to B$ and $g{:}B\to C$, is the arrow, $g\circ f{:}\ A\to C$.

This composition obeys the following two axioms:
- Identity: For every object *X*, there exists an arrow $1_X : X \to X$ called the identity arrow for *X*, such that for every arrow $f{:}A\to B$, it follows that $f\circ 1_A = f$ and $1_B\circ f = f$. Each object, *A*, has an identity arrow (1_A) with source *A* and target *A*.
- Associativity: If $f{:}A\to B$, $g{:}B\to C$, $h{:}C\to D$, then $(h\circ g)\circ f = h\circ(g\circ f)$.

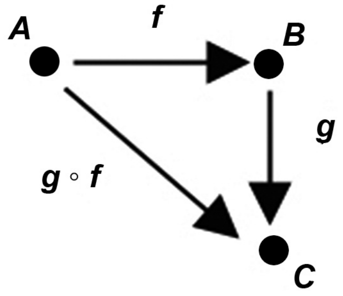

Fig. 1. A category with objects *A*, *B*, and *C* and arrows *f*, *g*, *g*∘*f*, and three identity arrows (not shown)

Examples

As noted, the ability to compose arrows in a category is a key property. So too is the realization that objects and arrows might represent almost anything.

URLs and QR codes

As one example, consider a class of literacy objects: anything displayed on paper, an electronic device, or on anything else that can be read from. Consider the arrows as the action of going from one object to another, be that by traditional reading, scanning, clicking on text, and so forth. When QR (Quick Response) codes first commonly appeared on the scene, occupying space on posters on subways, displayed in department store windows, or on hamburger wrappers in fast-food chains, many members of the public were left in the dark: what were these mysterious looking patterns of black and white checkerboard squares? Yet this same set of people probably had no difficulty clicking on a hypertext link to go to a website. They did not see the equivalence of structure (or had not had it explained to them). Viewed as category diagrams, the equivalence is clear (Figs 2 and 3): just substitute ″smart phone″ for ″computer″. Each has advantages: the mobility associated with the QR code does appear stunning at first.

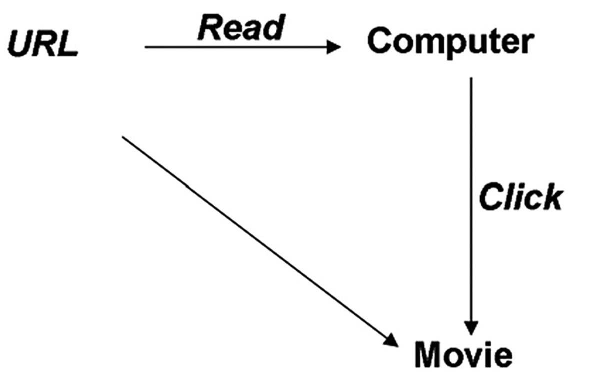

Fig. 2. URL to movie, as objects and arrows in a category

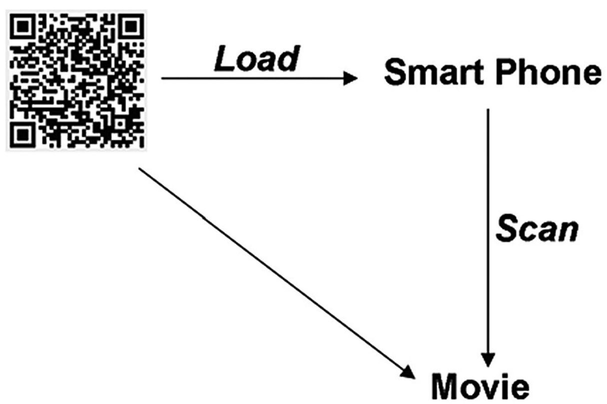

Fig. 3. QR code to movie, as objects and arrows in a category

Hyperlinks, as they might appear in various forms, URLs, QR codes, or embedded in other ways as yet to be imagined, are the backbone of the Internet. Dynamic connections have changed our world of communication.

In this example, it is generally clear to the user in either case that there is a two-step (or multiple step) process that led to the composition arrow of going directly from the target object to the source object. The next example shows a situation in which that staging is not necessarily as clear.

Thiessen polygons in the GIS software category

Our mapping world has changed from a static to a dynamic one. In the world of the past, when the population of a country changed, we drafted a new map. We no longer live in a world of India ink, Leroy lettering sets, pantographs, T-squares, Zip-A-Tone, and a whole host of fascinating cartographic hardware. It used to be that sloppy lettering on a map, or a smudged neat line, would require us to throw the map away and start over; a tedious process and one that was prone to drafting and other error. Now, the map and underlying data set are interactive: a change in one produces a change in the other. When the population of a country changes, we simply make the single change and are able to preserve earlier work. The corresponding shading and data ranges are adjusted and the new map is complete in a matter of seconds. As the digital map offers many opportunities, it is not a ″perfect″ replacement. Opportunity for error is abundant, although different from what it once was. One problem now is that a digital map in the hands of someone not well-versed in the conceptual background may lead to deceptive, misleading, or even dangerous results. The Internet is filled with example; one particularly famous one involves the creation of a map centered on North Korea in which error resulting from projection selection would have the reader believe, contrary to actual fact, that some parts of the world were within range of North Korean missiles, while others were not (*The Economist* 2003; Lightfoot 2003).

GIS software offers fine opportunity for executing complicated analysis in a straightforward manner. Even to a casual user, it is clear that a single button, tool, or wizard initiates and carries out the analysis. Contemporary software permits this sort of straightforward method for producing Thiessen polygons, based on perpendicular bisectors of lines joining dots within a dot scatter (Brassel, Reif 1979; Rhynsburger 1973; Kopec 1963; Thiessen, Alter 1911; Voronoi 1908). Some have claimed this feature as a ″new″ feature in more recent software packages.

In reality, though, it is a simple (although perhaps not straightforward) process to find such polygons without the facilitated straightforward approach. All that is really needed is basic knowledge of Euclidean geometry and of simple, standard GIS tools. Traditionally one might have used a drawing compass and a straightedge to construct a perpendicular bisector between two points (Coxeter 1961). If there are more than two points, the matter can become quickly tedious. The GIS offers a quick and accurate way to calculate positions:

- Given a distribution of points in the plane;
- Create circular, evenly-spaced, buffers around all the points, leaving the entire circle surrounding each point;
- The intersections of similarly sized circles will show positions for perpendicular bisectors of line segments (this is the standard construction from Euclidean geometry);
- It might be difficult to visualize the location of the set of perpendicular bisectors that are determined by this circular mass;
- Dissolve arcs within the circular mass. This procedure offers some help in visualizing where bisectors might be;
- To actually position the lines of partition, or Thiessen polygon edges, use the ″split polygon″ tool.

Now the process is complete (see Arlinghaus 2001 for an animation that shows the process). A quick visualization of the process might use a diagram such as the ones above, with a single arrow representing composition, from a source object of dot scatter to a target object of a partition of the plane by Thiessen polygons. The single arrow composition is formed from the steps above, from intervening stages, involving buffer, dissolve, and split polygon actions to move in stages from the source object to the target object.

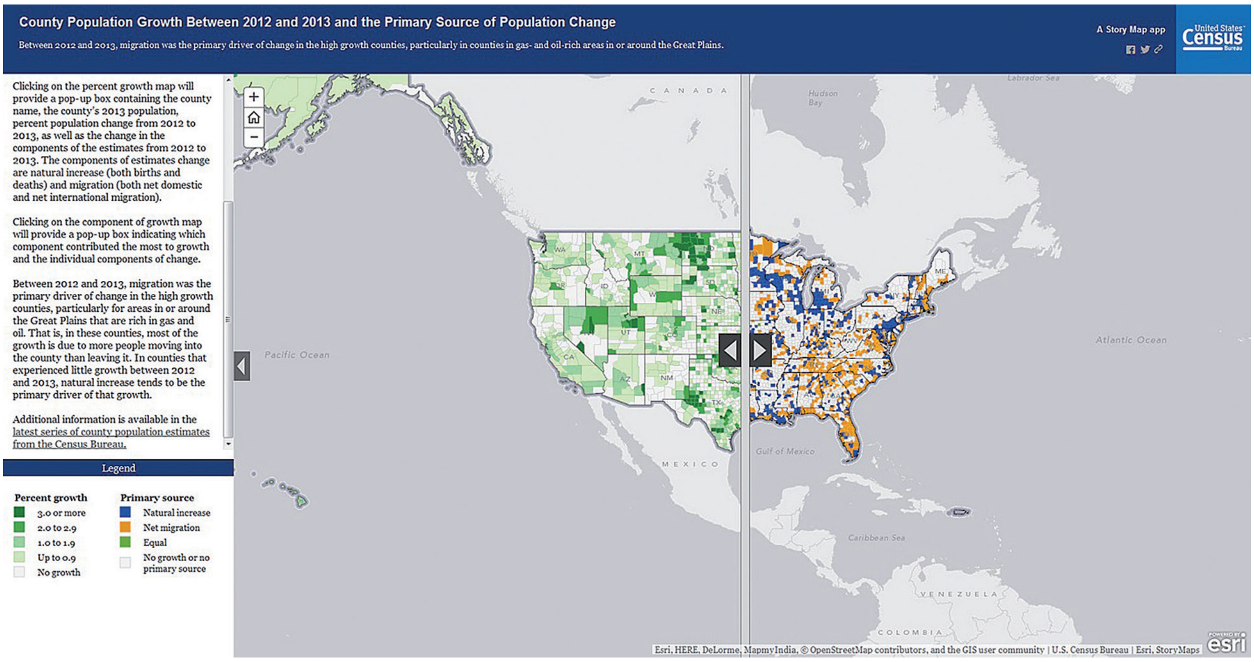

Fig. 4. Default view of mapping tool from US Census Bureau. Note the vertical gap in the center. The center gap slides and pattern changes accordingly.
Source: US Census Bureau and Esri.

A clickable map with a slider

A particularly interesting clickable map using the Esri story-map two-panel map template has appeared recently on the site of the U.S. Census Bureau (2014). It tells the story of population change over time and it does so with all files already resident on a server. When the map comes up in the browser, the default view is as it appears in Fig. 4. Note the visual vertical gap in the center; this gap is important as it carries the click-

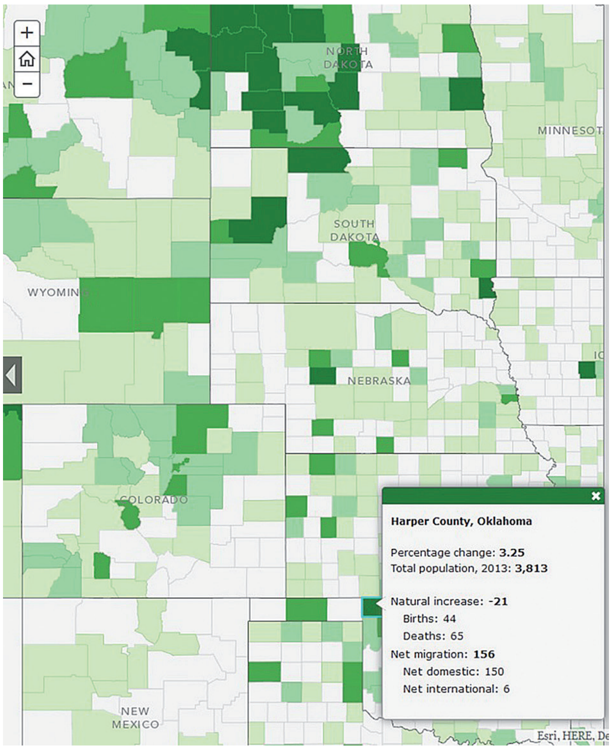

Fig. 5. Style of pop-up that appears when clicking on a county in the green area.
Source: US Census Bureau and Esri.

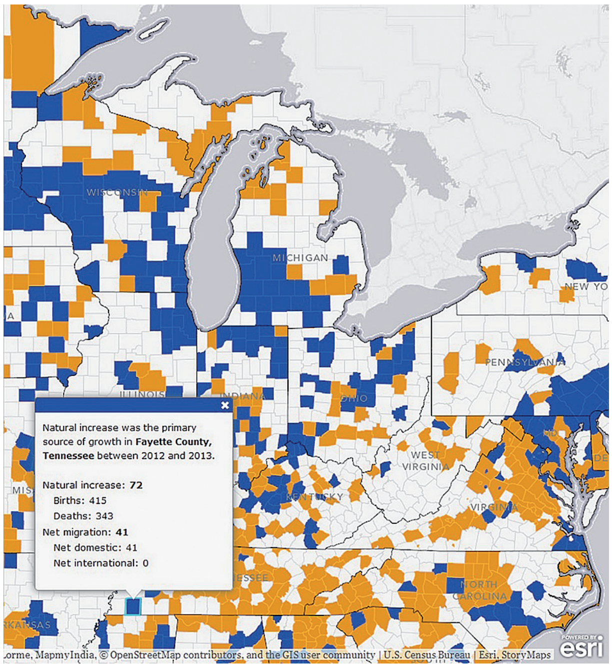

Fig. 6. Style of pop-up that appears when clicking on a county in the multi-colored area.
Source: US Census Bureau and Esri.

able map to a new level. Simple experimentation with clicking on the map will show that different pop-ups appear when one clicks on different sides of the gap. Fig. 5 shows a close-up of one for a county in the green pattern, while Fig. 6 shows another for a county in the multi-colored pattern. The explanation along the side of the map tells the reader of the differences: in the left side of the map in green tones, the leading lines of the pop-up focus on percentage change in population; in the right side of the map, in multiple colors, the leading lines of the pop-up focus on natural increase. Different map coloring patterns lead to target pieces of the underlying database that are identical in basic content but are arranged with different emphases and summary data.

Some readers of the map in Fig. 4 might have difficulty figuring out how to use the slider that is represented by the vertical gap. The process becomes clear when the map, data set, and motions associated with the map are viewed in a category diagram. The objects are the elements of the map and of its associated data set. Note that the designation of category objects in the world of pre-GIS mapping would have been different: spatial entities (such as polygons) and data would not have been viewed as interchangeable objects and therefore not grouped together in the same object

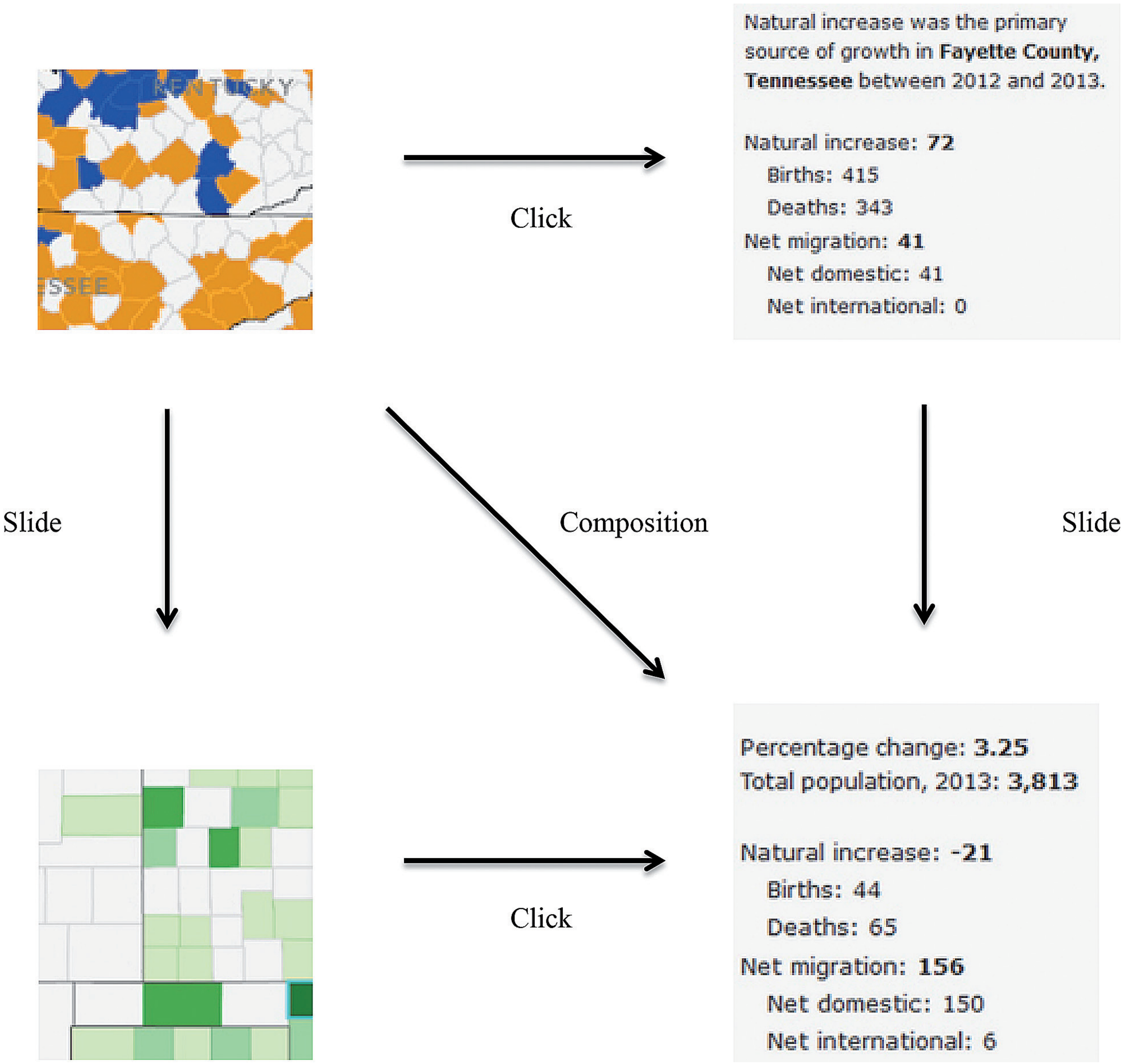

Fig. 7. Category diagram: arrows carry objects to other objects

class. The arrows are the motions associated with this map: "slide," the motion of the vertical gap, and "click," the motion to retrieve data from the underlying set. Fig. 7 displays the situation. The vertical gap in the census map permits the composition of arrows.

Diagram chasing

We used diagrams to follow the paths of arrows through relatively simple mazes of concepts. In more complicated systems, one way to visualize such sequences is to use a "commutative diagram". This structure is composed of objects and arrows in such a way that all directed paths in the diagram with the same source lead to the same target result by composition. Loosely, commutative diagrams are to category theory as equations are to algebra (Barr, Wells 1998). Commutative diagrams, and the associated "diagram chasing" of tracing the paths of elements around the diagram, serve as a method of mathematical proof involving numerous complexities of the arrows and objects. Fig. 8 shows a general commutative diagram; in it, the sequence of arrows from the source X to the target Z that goes along the top and right yields the same result as the sequence of arrows that goes along the left and the bottom: f then g (denoted $g \circ f$) leads to the same target as h then k (denoted $k \circ h$). The diagram commutes. So too, the map diagram in Fig. 7, involving the Census map, also commutes.

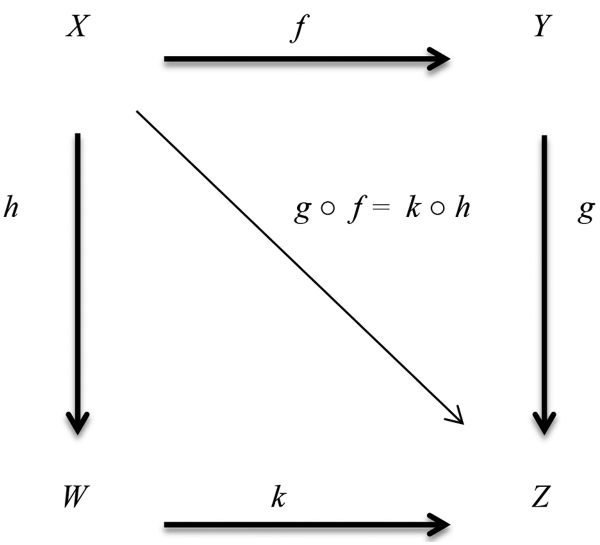

Fig. 8. General simple commutative diagram. Arrows carry objects to objects

The diagrams presented up until now have been ones with arrows, objects, and a binary operation that satisfy conditions to be a category. Consider, however, the routine classroom diagram in Fig. 9. It does not represent a category; diagram chasing, of elements carried around the figure by arrows, will show that associativity does not hold here. Associativity fails: $(h \circ f) \circ g = 1_A \circ g = g$, while $h \circ (f \circ g) = h \circ 1_B = h$. Grouping of operations matters. If it held, one would expect the sequence of arrows to lead to the same result, and here it does not.

Commutative diagrams of greater complexity, used in proving intricate mathematical propositions, are abundant on the Internet. The interested reader might look at articles on the Five Lemma, for example (Wolfram 2014). Category theory abstracts the detail away from complicated systems in order to find meaningful underlying principles and proofs. Digital maps and their underlying huge data sets form a complicated system. The "story maps", of the sort created by Esri and the US Census Bureau, offer a lead on one idea of how to merge maps and math (Fig. 7).

Proofs and results may follow more easily, and simply, when mathematical structure is cast in the broader category framework. As Blass notes (1984: 8–9), "It is a remarkable empirical fact that the important structural properties of mathematical objects are often expressible in category-theoretic terms, specifically as universal properties. (...) Not only do universal descriptions exist, but they are useful in at least two ways. First, they tend to express the more important properties of mathematical structures, so that keeping them in mind helps one to avoid irrelevant complications. (…) second (…) they can reveal structural similarities between concepts in diverse areas of mathematics".

Any manner of formalizing a mathematical concept such that it meets the basic category-the-

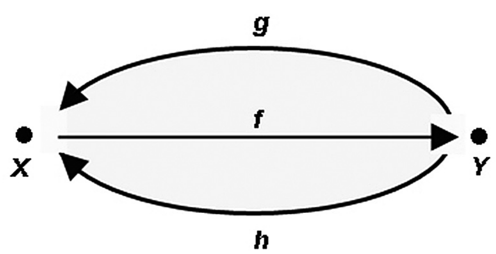

Fig. 9. This diagram of arrows and objects does not represent a category. Associativity fails

oretic conditions on the behavior of objects and arrows is a valid category, and all the results of category theory will apply to it.

Indeed, categories now appear in most branches of mathematics, as well as in some areas of theoretical computer science (Barr, Wells 1998). In addition to standard scientific references, one blog suggests the use of category theory for developing better spreadsheets (Baez 2014). Another tracks comments about category theory and its use in the sciences and elsewhere (The n-Category Café, March 3, 2013). While Baez (2006) notes, echoing sentiments of mathematician Blass, that "Faced with the great challenge of reconciling general relativity and quantum theory, it is difficult to know just how deeply we need to rethink basic concepts. By now it is almost a truism that the project of quantizing gravity may force us to modify our ideas about spacetime. Could it also force us to modify our ideas about the quantum? So far this thought has appealed mainly to those who feel uneasy about quantum theory and hope to replace it by something that makes more sense. The problem is that the success and elegance of quantum theory make it hard to imagine promising replacements. Here I would like to propose another possibility, namely that *quantum theory will make more sense when regarded as part of a theory of spacetime*. Furthermore, I claim that *we can only see this from a category-theoretic perspective* – in particular, one that de-emphasizes the primary role of the category of sets and functions". Might such an approach lend similar power and opportunity to the science of geography?

Category theory: universal concepts and GIS software

Simple views of familiar spatial objects such as maps, coupled with the observations of practitioners in other disciplines, suggest that category theory might see a deep use in the field of geography. For example, if one can cast GIS functionalities in places (such as staying up to date with GIS; tracking change in GIS over time; and so forth) as a category based on universal concepts defining GIS software, then tracking change over time as software improves might substantially ease the learning curve presently involved in staying up to date with GIS. We saw one example involving the idea of Thiessen polygons and how calculation of those had evolved through different variations of software. Imagine capturing each entire GIS version as a single category diagram and then mapping one category to another to track change in GIS over time using the universal concepts as benchmarks. In such an approach, it is clear that the identification of universal concepts is critical.

Albrecht (1999) and others have attempted to create sets of universal GIS concepts and they note the importance of doing so; indeed, Albrecht used a Stella-type diagram to attempt to visually track the concepts. We suggest that category theory, with its general and highly abstract language, might be particularly well-suited to this task. To implement this large project, we propose to begin by creating an archive of universal GIS concepts in relation to older software and move forward through the spectrum of new software versions to understand how one category maps to another. The project will evolve as work proceeds; the general pattern, however, is to view a piece of GIS software, complete with data sets, as a single category. With those in hand, a category of categories should emerge to capture the entire structure in a unified view. From that vantage point, both research and educational agendas might proceed in a variety of unforeseen directions.

To generate sets of useful universal concepts, the authors invite interaction and suggestions from others, in the spirit of *Quaestiones Geographicae*!

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