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Some fundamental differences between Christaller $K = 3, 4$, and 7 systems: spatial autocorrelation at work

Abstract: This essay highlights mathematically derived and geographically grounded differences in the properties between Christaller's three canonical central place schemes ($K = 3, 4$, and 7), with special emphasis on spatial autocorrelation (SA). It reveals the following groundbreaking insights: (1) implications from judiciously augmented increases in the minimum geographic landscape size that accompany displaying a six-tier hierarchy across the three K values; (2) contiguous SA dominates hierarchical SA; (3) negative SA reigns; (4) SA variance inflation in population counts is considerable, masking the Poisson random variable nature of a system of cities' populations; and, (5) much can be gained by studying geographic landscapes that are substantially larger than those portrayed in textbooks and almost all research publications. Local indicators of spatial association (LISA) and Griffith-Moran eigenvector spatial filtering (GMESF) analyses confirm systematic differences in both local and global SA within these separate ideal geographic landscapes. Number-theoretically, $K = 3, 4$, and 7 serve as canonical prime generators of many higher-order Löschian numbers. These distinctions provide a robust, multidimensional SA-anchored framework for characterizing and discriminating among central place hierarchies, suggesting broader potential for integrating number theory and other mathematical subdisciplines with spatial analysis in settlement research.

Keywords: central place theory, Christaller, Griffith-Moran eigenvector spatial filtering, spatial autocorrelation, sui-Poisson

Introduction

Christaller (1933) and Lösch (1944) are the well-recognized central place theory (CPT) architects (see Appendix A) who posited the following: human settlement networks form a hierarchical system of relatively centrally positioned cities that supply goods/commodities/services to their surrounding regular hexagonal-shaped hinterland market areas, with each level distinguished by the

threshold and range of its offered subset of goods/commodities/services, and with a given system indexed by its nesting ratio (i.e., what they dubbed a K value) specifying how many lower-order centers fall within the market area of each of its next-higher-order ones. Intriguingly, $K = 3$ denotes marketing optimality, $K = 4$ denotes transport efficiency, and $K = 7$ denotes administrative effectiveness. Christaller initially formulated this conceptualization based upon these three preceding principles, with Lösch later generalizing it to an infinite family of possible nesting ratios, many of which are combinations of the $K = 3, 4$, and/or 7 landscape organizations – implanting a number theory standpoint to it. Their fascination with this topic also has mesmerized other segments of society within and outside of the official social sciences, some even in the full academy itself (e.g. Isard 1956, Beckman 1968, Gould 1969, Smith 1979, Hayes 2004, Fulminante 2012, Shi et al. 2020, Eniang 2025).

The purpose of this paper is to highlight mathematically derived and geographically grounded differences in properties between the three canonical central place (CP) schemes, some being newly discovered mindsets (e.g., see Kaczorowski et al. 2024, Griffith 2026a), all the while keeping spatial autocorrelation (SA) in the foreground. Noteworthy from the start are similarities such as the SA mechanisms of interest being positive-negative mixtures exhibiting a dominant negative SA component, adding to the growing literature demonstrating that this blended variety truly constitutes that for the majority of georeferenced data. Inspection of SA-informed – both contiguous and hierarchical – milieus ranging from ones exceeding the largest, but still a quite small, number of settlements usually portrayed in pedagogic and other academic publications, to the even larger number that barely retains, rather than adding an additional echelon to, Christaller's default six-tier pyramids¹, constitutes the geospatial testbed for this discussion.

The geographic scale spectrum: the six-tier system range

One important differentiating trait, which constitutes new knowledge, is the failure of the human settlement populations in a $K = 3$ (i.e., the market principle) system to simplify easily from a negative binomial (NB) to a Poisson random variable (see Table 1; Griffith 2026b). The most frequently visualized $K = 3$ system, vis-à-vis typical $K = 4$ and 7 depictions, endures the unique positive-SA-suppressing facet of relatively few lowest-tier settlements being first-order neighbors, with this layer becoming more positive-SA-friendly when the bottom urban hierarchy strata collapse to a single level as n goes beyond 919, paralleling the real-world taxonomy complications affiliated with dwarf/country towns, villages, and hamlets. Increasing the nesting factor K to larger numbers (e.g., 4 and 7 , or even Löschian numbers such as 13 and 19) achieves this same degree of

¹ Dwarf, country, and small and medium-sized town, city, and metropolis (p. 59). This adopted count is an arithmetic average of levels presented in graphics (e.g., pp. 66, 73, 75) and tabular information (e.g., p. 67): $(5 + 7) / 2 = 6$.

juxtaposed-cities similarity effect. Including an accounting for hierarchical SA (Griffith et al. 2025) rectifies this $K = 3$ fallacy. In other words, the sui-NB (e.g., Griffith 2024) dispersion parameter is exactly zero, eliminating extra-Poisson

Table 1. $K = 3$ settlement frequency distributions by hierarchical level

level	n = 271	n = 331	n = 397	n = 469	n = 547	n = 631	n = 721	n = 817	n = 919
1	180	222	264	312	366	420	480	546	612
2	54	72	96	96	120	150	150	180	216
3	24	24	24	42	42	42	60	60	60
4	6	6	6	12	12	12	24	24	24
5	6	6	6	6	6	6	6	6	6
6	1	1	1	1	1	1	1	1	1
$1^T C_c 1$	1512	1860	2244	2664	3120	3612	4140	4704	5304
vectors [†]	3 + 5	3 + 3	1 + 3	6 + 10	8 + 9	4 + 5	10 + 11	7 + 9	8 + 9
$1^T C_h 1$	2712	3276	3984	4812	5616	6648	7836	8916	10236
vectors [†]	1 + 2	1 + 4	1 + 5	2 + 0	2 + 2	4 + 6	2 + 4	4 + 0	3 + 2
$\cap$	1008	1236	1500	1776	2076	2412	2760	3132	3540
$\hat{\eta}$	1.7702	1.6369	1.5010	1.4753	1.3781	1.2807	1.3311	1.2555	1.1799
$\hat{\eta}_{c_c}^*$	0.4270	0.2245	0.1563	0.3280	0.2262	0.1815	0.2801	0.2133	0.1492
$\hat{\eta}^*$	0	0	0	0	0	0	0	0	0

Note: $1^T C_c 1$ and $1^T C_h 1$ respectively denote the sum of ones in their embedded contiguous and hierarchical spatial weights matrices, with the intersection symbol $\cap$ denoting the shared ones occurring in these two matrices; $\hat{\eta}$, $\hat{\eta}_{c_c}^*$, and $\hat{\eta}^*$ respectively denote the raw, contagion-SA-adjusted, and SA-adjusted negative binomial (NB) dispersion parameter estimates.

[†] stepwise selected positive plus negative SA GMESF vectors count.

Source: own elaboration.

Table 2. $K = 4$ and 7 settlement frequency distribution by hierarchical level

level	$K = 4$				$K = 7$			
	n = 817	n = 1387	n = 2107	n = 2977	n = 7351	n = 7651	n = 7957	n = 8269
1	600	1056	1560	2256	6300	6558	6822	7086
2	156	240	420	552	882	924	966	1014
3					144	144	144	144
4	54	84	120	162	18	18	18	18
5	6	6	6	6	6	6	6	6
6	1	1	1	1	1	1	1	1
$1^T C_c 1$	4704	8064	12324	17484	43512	45300	47124	48984
vectors [†]	4 + 4	7 + 22	10 + 13	16 + 29	8 + 38	11 + 43	12 + 47	10 + 48
$1^T C_h 1$	7128	12120	19416	27492	67020	69540	72252	75192
vectors [†]	2 + 4	2 + 0	3 + 0	2 + 0	3 + 2	1 + 2	4 + 2	5 + 1
$\cap$	2400	3972	6240	8652	12432	12948	13452	14004
$\hat{\eta}$	1.5252	1.3737	1.2426	1.1850	0.3290	0.3169	0.3054	0.2943
$\hat{\eta}_{c_c}^*$	0	0	0	0	0	0	0	0
$\hat{\eta}^*$	0	0	0	0	0	0	0	0

Note: see Table 1 footnotes.

Source: own elaboration.

variation, hence streamlining the probability model description of CP populations to an undisputed sui-Poisson one. Griffith et al. (2019) describe how to formulate and execute the accompanying Griffith-Moran eigenvector spatial filtering (GMESF) results reported in Table 1 (as well as its ensuing Table 2). Appendix B describes how to count the level-specific CPs, and create the affiliated map (e.g., Fig. 1).

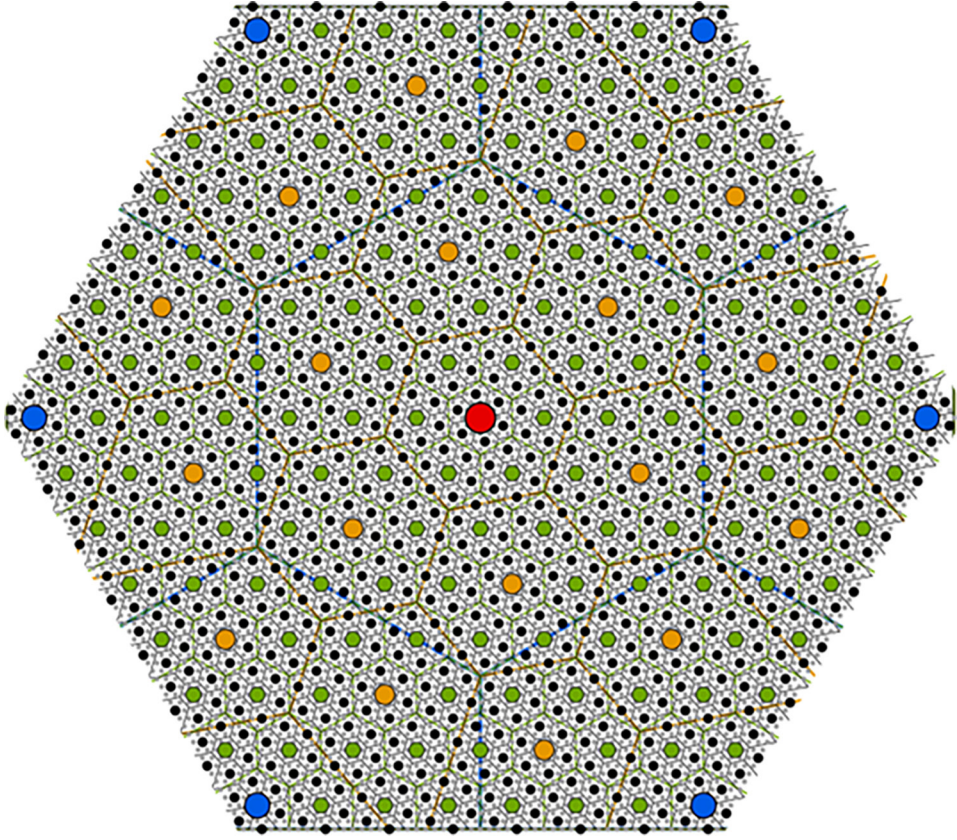


Fig. 1. A six-tier $K = 7$ Christaller CP system; $n = 8,269$. Legend: hamlet (gray), village (black), small town (green), medium-sized town (orange), city (blue), metropolis (red)

Source: own elaboration.

An already infamous, although not well-disseminated, Christallerian CP complex aspect is the $K = 4$ (i.e., the transportation principle) system failure to include a third level. Its two-dimensional configuration yields a hierarchy in which this third tier is a geometric impossibility². Because the general spatial expansion

² Central place hierarchies build upon a triangular grid, and hence correspond to Eisenstein integer sublattices, where hierarchical levels arise through algebraic dilation of a lattice. When this dilation by rational integers producing only axis-aligned scaling fails to yield a new central place

factor between consecutive hierarchical echelons is $\sqrt{K}$, the square root of the prevailing nesting ratio, the transport principle imposes a strict doubling of inter-center spacing (i.e., $\sqrt{4} = 2$). The first doubling generates second-order centers located midway along transport routes between higher-order places. However, the second doubling places the prospective third-order centers exactly on the lattice nodes already occupied by first-order centers. Geometrically, these hypothetical intermediate positions coincide with this system's highest-order positions, eliminating the spatial foundation for a distinct third level. Functionally speaking, the operating transport efficiency objective reinforces this strata collapse: any attempt to demarcate intermediate hinterlands would produce redundant concomitances along primary transport corridors. Accordingly, the absent layer of third-category cities is an inherent mathematical property of the $K = 4$ hierarchy, not a mistake or an empirical anomaly.

Distinguishing it from $K = 3$ and 4 , $K = 7$ is mathematically unique because it forms fully nested, non-overlapping hexagonal triangular vertices (but not their collective hinterlands) at every hierarchical level, with each higher-order center supervising exactly six subordinate sites, guaranteeing a complete, gapless hierarchy. Erecting this better balancing upon abstract algebra cognition terms³, $K = 7$ is like a perfect honeycomb arrangement of cities: every bigger city sits at the center of a hexagon of six smaller cities, and each of those smaller cities does the same, at every hierarchical level and geographic scale. In other words, whether zooming in or out, the map pattern looks the same. Consequently, $K = 7$ is the most fractal-like of Christaller's three canonical systems.

Tables 1 and 2 insinuate that the extra-Poisson dispersion engendered by SA variance inflation tends to decline with increasing geographic landscape size as well as K value. The contiguity SA eigenvectors suggest that the nature of their source correlation becomes more complex as a CP system enlarges and K rises. Furthermore, any overlap between the contiguous and hierarchical spatial weights matrices tends to decline as K gets larger.

Discussion

In keeping with the goal of highlighting mathematically derived and geographically grounded differences in the inimitable properties varying across the three Christallerian canonical schemes, this section first expounds on the discussion thus far. Next, it branches out from a global SA preoccupation to locally focused SA. Third, it engages in a number theory (e.g., Barnett 1969) relevancy

hexagonal configuration, these intermediate levels that are structurally present also are spatially unoccupied, accounting for the universally missing third echelon in Christaller $K = 4$ systems.

³ For example, a rank-2 free abelian module over $\mathbb{Z}$ (i.e., all integers, both positive and negative, providing the modular arithmetic necessary to identify the sublattices comprising specific hierarchy level cities) with a descending 7-adic filtration, whose quotient at each level is $\mathbb{Z}_7$, instilling a perfect hexagonal nesting of central places.

reconnaissance. Not surprisingly, then, its aim is to bolster evidence already garnered in this paper spotlighting exclusivity in the prototype $K = 3, 4$, and 7 CP creations.

Geometry- and global-SA-anchored discriminators

Tables 1 and 2 disclose the following groundbreaking revelations: (1) large increases in the minimum geographic landscape size accompany the feasibility of displaying a six-tier hierarchy across the three K values (i.e., $n = 271 \rightarrow 817 \rightarrow 7,351$); (2) contiguous SA dominates hierarchical SA; (3) negative SA reigns – which should be intuitively obvious from a visual perusal of any multiple CPs network map; (4) SA population counts variance inflation is considerable; and, (5) much can be gained by studying geographic landscapes that are substantially bigger than those portrayed in textbooks and other published scholarly outlets. The most obvious mathematical differences come from converting a regular square mesh integer lattice $\mathbb{Z}^2$ with coordinates (u, v) into a triangular lattice web via modular arithmetic. The following identifies the sublattices in $\mathbb{Z}^2$ that undergo this Eisenstein transformation – $u^* = u - v/2$, and $v^* = v(\sqrt{3})/2$ – spawning CP hexagons: for hierarchical level $h \geq 1$,

marketing principle ($K = 3$): $u \equiv 0 \pmod{3^{\lfloor (h-1)/2 \rfloor}}$ and $v \equiv 0 \pmod{3^{\lfloor (h-1)/2 \rfloor}}$,

where the superscript denotes the integer-only part of $(h - 1) / 2$

transportation principle ($K = 4$): $u \equiv 0 \pmod{2^{h-1}}$ and $v \equiv 0 \pmod{2^{h-1}}$

administrative principle ($K = 7$): $2u - v \equiv 0 \pmod{7^h}$ and $u + 3v \equiv 0 \pmod{7^h}$, even levels; $u \equiv 0 \pmod{7^h}$ and $v \equiv 0 \pmod{7^h}$, odd levels with $h > 1$.

Moreover, clear mathematical distinctions separate these three notions. Therefore, without devoting proper thoughtfulness to these underlying peculiarities, these three systems can appear superficially similar, even though they operate through fundamentally different geometric and algebraic mechanisms.

Meanwhile, understanding how SA operates within Christaller's CP systems, a newly minted research endeavor, requires close attention to the geometric and arithmetic structures underlying the $K = 3$ (marketing), 4 (transport), and 7 (administrative) principles. Although these complexes and their accompanying city networks share a common hexagonal spatial logic, the propagation of similarity across space – the foundational essence of SA – differs systematically across them because each K value imposes a distinct pattern of hierarchical nesting and connectivity, neighbor proximity among peer cities, and spatial interaction flow constraints. In the $K = 3$ system, market-catchment overlap, and hence territorial competition, primarily governs SA. Each service center's tributary area touches six others through relatively short interaction paths, producing high first-order

neighbor similarity and rapid spatial sharing of attribute redundancies. As a result, positive SA is strong but shallow: clustering is pronounced, yet similarity rapidly attenuates with hierarchy. This concoction produces a spatial structure in which local interactions dominate, and higher-order CPs exert only modest influence beyond their immediate rings of lower-order places.

The $K = 4$ transport system recalibrates these interaction pathways. Because centers now align along optimized transport channels, similarity propagates preferentially along linear passageways as opposed to uniformly across a homogeneous and isotropic – as well as, suppositionally, unbounded and continuous – plane housing them. This directional bias introduces corridor-oriented anisotropic SA: similarity remains strong, but its decay varies by direction. Map patterns consequently take on an axial, perhaps even radial, like coherence, with long chains of similar places forming along major routes, coalescing amid weaker spatial dependence/correlation orthogonal to them. The result is a geographic field in which distance alone no longer predicts SA strength; the prevailing travel network geometry becomes the overriding organizing force.

Finally, the $K = 7$ administrative system exhibits a markedly different SA behavior. Here, each higher-order center fully contains a set of immediately lower-order settlements in a strict, non-overlapping nesting structure for these discrete hinterland-midpoint entities, although parts of their borderlands as well as deeper nested places may be mutual (Fig. 1). This geospatial organization produces a deep hierarchy in which spatial similarity intensely propagates across multiple geographic scales through parent-child administrative descent instead of mere contiguity. The outcome is a much more robust SA structure: attribute values at even distant locations may remain strongly correlated if they fall within the same administrative lineage, whereas adjacent places belonging to different lineages tend to exhibit sharper discontinuities. Such attachment linkages forge a multilevel, quasi-fractal pattern of spatial dependence/correlation, one in which similarity strengthens not by proximity but by hierarchical encapsulation.

Taken together, the three Christaller systems demonstrate that SA is not simply a by-product of distance decay; rather, it is a structural aftermath of the joint geometric, functional, and administrative logics embedded in CP organization. Each K value imposes a distinctive set of constraints on how similarity flows across geographic space, emphasizing: contiguous market overlap for $K = 3$, directional transport pathways for $K = 4$, and telescoping hierarchical containment for $K = 7$. These unmistakable differences reveal that classical CPT implicitly encodes three mathematically disparate modes of spatial dependence/correlation – local, anisotropic, and hierarchical – each with helpful ramifications shedding light on how to interpret empirical settlement patterns. As Tables 1 and 2 corroborate, GMESF and its affiliated sui- random variable conceptualization furnish invaluable spatial analytic tools for pursuing this type of explanation.

Local SA discriminators

Tables 1 and 2 confine contemplative responsiveness to six-level pyramids, which both their $K = 3$ and 4 systems challenge. Fine-tuning Christaller’s vision allows a moderate settlement-nodes typology refinement: four-corners community, hamlet, village; small, medium, and large town; minor, mid-size, and big city; and, miniature and intermediate metropolis, and megalopolis. The 2020 United States (US) Census reports roughly 32,188 such incorporated and census-designated places across its core and insular territories, approximately four times greater than the largest $K = 7$ entry in Table 2. China’s 2020 census inventoried administrative cities and towns numbering about 21,533, with several hundred qualifying as mega-cities having much larger populations than their US counterparts [e.g., Beijing, Chongqing, and Shanghai versus New York City (NYC)]. Again, these amalgamations are many times greater than the largest $K = 7$ entry in Table 2. India, the only other country having more people than the US, has 7,935 cities and towns enumerated in its latest, 2011, census – today’s count most likely is closer to 9,000 – with Dehli and Mumbai having larger populations than NYC. The India situation is closer to the order of magnitude for the largest $K = 7$ entry in Table 2. Thus, although Figure 1 symbolizes the largest CP settlement image now freely available and publicly pictured to date, particularly for a $K = 7$ system, much larger ones warrant examination. The $K = 3, 4$, and 7 specimens investigated in this section have $n = 32,137$ and six to 10 tiers⁴.

Tables 1 and 2 document the presence of negative SA-dominated positive-negative mixtures in this paper’s experimental CP geographic distributions of population. Although GMESF detects this subtly, it is not necessarily sensed by local indices of spatial association⁵ (LISA; Anslin 1995). The following, corroborating this mixture contention, is a cross-tabulation of statistically significant LISA for Figure 2; these results are obtainable with the ArcGIS Pro spatial statistics toolbox [i.e., Cluster and Outlier Analysis (Anselin Local Moran’s I)]:

K; figure	high-low (HL)	low-high (LH)	high-high (HH)	low-low (LL)
3; 2a	3121	1727	80	5
4; 2b	1545	1306	0	0
7; 2c	4591	162	0	0

⁴ tier	1	2	3	4	5	6	7	8	9	10
K = 3 settlement frequency	21426	7140	2382	792	264	96	24	6	6	1
K = 4 settlement frequency	24180	6006	0	1824	90	30	6	1		
K = 7 settlement frequency	27546	3960	540	72	18	1				

⁵ LISA is a spatial statistical quantifier of local SA patterns in georeferenced data, a tool that allows researchers to move beyond map-wide global SA measures by examining, in turn, location-by-location variations in map patterns. Importantly, LISA provide insights into local spatial relationships, which global measures can camouflage through their arithmetic averaging. They identify geographic concentrations of high and low attribute values: high values surrounded by high (HH) or low (HL) values; and, low values surrounded by low (LL) or high (LH) values.

Figure 2 portrays the statistically significant LISA high-low (HL) contrasts of the plurality of these couplings. Its visual appearance, together with these numerical tallies, confirms conspicuous pairwise deviations across these three CP systems. These numerical repercussions are consistent with theoretical expectations, given that so many lowest- and higher-level settlement hinterlands abut, while few-to-a-modest-percentage of the lowest levels also have laterally juxtaposed market areas. An overwhelming number of local relationships are negative SA ones (i.e., HL and LH). $K = 3$ LISA mirror its prevailing positive-negative SA blending. The shift to almost nothing but HL in $K = 7$ underscores its administrative principle of nesting in which larger population cities are perfectly encircled by the smallest population settlements. Finally, relative positionings activate the perceptible spacing variations between local clusters Figure 2 highlights.

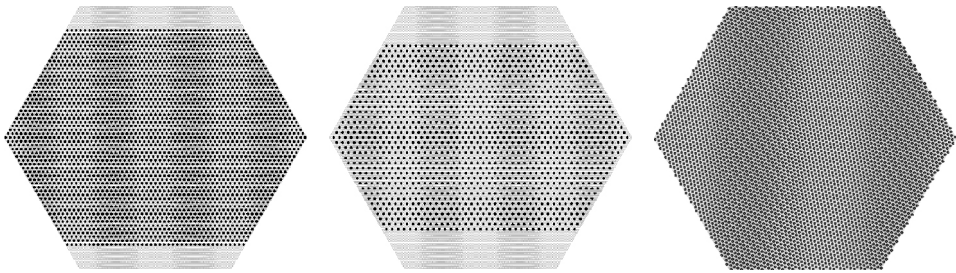


Fig. 2. The $n = 32,137$ Christallerian CP geographic landscape with superimposed statistically significant LISA HL geographic distributions. Left (a): $K = 3$. Middle (b): $K = 4$. Right (c): $K = 7$

Source: own elaboration.

This LISA exercise argues for the capability of such spatial statistical tools to divulge suitably unique particularities isolating each of the three canonical CP systems. In addition, the relative prevalence of HL/LH contrasts corresponds to the smallest and largest tiers being side-by-side, whereas HH/LL contrasts are rare (i.e., just a paucity strictly in $K = 3$) because similar-tier settlement populations rarely adjoin in these three and other such idealized lattices.

Number theory discriminators

While Christaller elucidated the $K = 3, 4$, and 7 archetypal systems, Lösch extended the settlement hierarchy nesting idea to an infinite number of supplemental possibilities, some of which are convolutions (i.e., powers) of these three individual numbers, their pairings (i.e., cross-products of them and/or their powers), and their triad (i.e., a cross-product of all three, sometimes involving powers). Dacey (1965) formulated the following Löschian K -value generating function based upon the integer lattice coordinates (u, v) prior to their Eisenstein transformation: $u^2 + uv + v^2$; $u = 1, 2, \dots$ & $v = 0, 1, 2, \dots$. Accordingly, one can define a Löschian number system in which the canonical CPT K values – 3, 4, and 7 – are treated as prime elements (i.e., each is dividable only by 1 and

itself). In this system, divisibility is interpreted in terms of hierarchical nesting: a number K is composite if it is expressible as a product of lower-order K values corresponding to subordinate settlement levels; otherwise, it is prime. By this definition, 3, 4, and 7 are irreducible because they generate the fundamental hexagonal hierarchies described by Lösch, and no smaller K values can combine to reproduce them. Non-canonical K values, such as 12 or 28, can be factored into products of these prime elements, reflecting higher-order or mixed hierarchical structures. This framework formalizes the algebraic relationships among selected K values, allowing their combinatorial and spatial properties to be analyzed in a consistent numerical system while preserving the conceptual logic of CP hierarchies.

Key definitions for this system include the following:

- DEF #1: L is the set of Löschian numbers (i.e., integers corresponding to K values in CP hierarchies).
- DEF #2: A prime number $p \in L$ is one that cannot be expressed as a product of smaller prime numbers in L .
- DEF #3: The canonical prime numbers in L are 3, 4, and 7 (i.e., the Christallerian K values).
- DEF #4: Divisibility $a|b$ means the construction of its corresponding hierarchy is by nesting a -level structures.

Given these preliminaries, a fundamental Löschian factorization theorem, which is about operations generating spatial structures rather than numbers, may be written as

THM #1: If L is the set of canonical Löschian numbers (i.e., those obtained solely by successive nestings of the Christaller $K = 3, 4$, and 7 systems), then every $K \in L$ admits a unique factorization such that

$$K = 3^\alpha 4^\beta 7^\gamma, \text{ with } \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}, \text{ and } \alpha + \beta + \gamma > 0.$$

PF: Each canonical nesting K value multiplies the existing system by exactly one of the three Christaller transformations with ratios 3, 4, or 7. After, respectively, a α , β , and γ application of the $K = 3, 4$, and 7 transformations, the resulting nesting ratio is $K = 3^\alpha 4^\beta 7^\gamma$. Conversely, any finite sequence of canonical nestings produces a value of this form. Uniqueness follows because this nesting process defines a free commutative monoid on the generators $\{3, 4, 7\}$: the exponents α , β , and γ record the number of applications of each transformation, and therefore are uniquely determined by the construction of K .

REMARKS: This foregoing factorization holds only for the canonical Christaller subfamily, and does not characterize the full set of Löschian numbers, which also includes primitive non-canonical values, such as 13, 19, and 25. These three atomic Christallerian systems generate an infinite but asymptotically negligible subset of this Löschian spectrum; although their count dominates early in the L sequence, almost all Löschian numbers arise from non-canonical primitive factors. Nevertheless, a differentiating feature among these Christallerian K values is the distinct lattice-scaling transformations each induces

on a hexagonal (triangular) lattice. In other words, this theorem documents that the three Christaller systems differ by their specific individual similarity transformation that they impose on a hexagonal lattice, with $K = 3, 4$, and 7 constituting the minimal, inequivalent generators of canonical hierarchical nesting. Notably, this proposition can be a corollary to more comprehensive theorems covering all L , emulating like performances by the fundamental theorem of arithmetic (e.g., Ireland, Cuoco 2023).

Non-canonical Löschian numbers arise from Eisenstein-integer norms not reducible to $3, 4$, or 7 , and cannot be expressed as products of canonical primes. Clearly, as Table 3 also attests, number theory is another mathematical dimension for segregating these canonical CP systems. Obviously, many more definitions, theorems, corollaries, and other formal propositions may be posited.

Table 3. Mathematical peculiarities of Christaller systems via number theory

property	$K = 3$	$K = 4$	$K = 7$	consequence/interpretation
primality/irreducibility	prime	prime	prime	unfactorable into smaller K values; fundamental hierarchy generators
distinct lattice generators	linearly independent	linearly independent	linearly independent	no power of one prime produces another
minimal lattice generator	yes	yes	yes	smallest integer producing valid hexagonal tiling for its principle
greatest common divisor (GCD)	mutually co-prime (GCD = 1)	mutually co-prime (GCD = 1)	mutually co-prime (GCD = 1)	ensures hierarchical lattices combine without redundancy
least common multiple (LCM)	84	84	84	the canonical-sourced minimal composite lattice; $3 \times 4 \times 7$
combinatorial significance	3 subordinates	4 subordinates	7 subordinates	the subsidiary settlement count in canonical hierarchies
factorization role	building block	building block	building block	products of these primes define one non-canonical K value set

Source: own elaboration.

Consequently, one takeaway is that the more general subdiscipline of number theory offers a rich setting for better articulating CPT. Prime numbers are important in mathematics (e.g., Curtis, Tularam 2011), and come in many notable varieties: Eisenstein, Fermat, Gaussian, Mersenne, Sophie Germain, twin, and rational, among many others. No doubt, Löschian primes⁶ (e.g., à la Alao et al.

⁶ In the 1st 100 Löschian numbers, the primes are: 3, 4, 7, 13, 19, 25, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139, 151, 163, 181, 193, 199, 211, 223, 229, 241, 271, 277, 283, 307, and 313. These are a subset of the natural primes in this interval (i.e., 1-343), with only 25 as an incongruity (i.e., because 5 is not a Löschian number).

1977, p. 43) are important, too, if for no other reason than to enable an augmenting of tenets crafting the geographic organization of settlement networks.

Dénouement for this section

In summary, Christaller's canonical $K = 3, 4$, and 7 CP systems are distinguishable across a myriad of suitable analytical criteria. Geometrically, and an already well-known graphical outcome, each K value imposes a unique hierarchical pattern on hexagonal lattices, producing distinct global and local SA intricacies. LISA and GMESF analyses reveal systematic local SA differences consistent with the posited Christallerian structural principles. Number-theoretically, $K = 3, 4$, and 7 act as prime generators of canonical Löschian numbers and their associated CP networks, while other prime-number-based systems remain to be interpreted in the context of increased settlement numbers, sizes, and peer-order spacings. All composite K values can be derived from this full complement. Non-canonical allied numbers, such as 13 or 19 , lying outside of this scheme, have yet to be linked to spatial organization principles. Together, these two collections provide a robust, multidimensional framework for discriminating amongst CP hierarchies, and suggest broader potential for integrating number theory and other mathematical subdisciplines (e.g., abstract algebra: under a lattice formulation in which settlement locations are elements of a translation group and higher-order centers define nested subgroups, the set of places tethered to a given center forms a coset of the corresponding subgroup) with spatial analysis in settlement research. Therefore, perhaps debatably, the contents of this section succeed in furnishing credible support illustratively accentuating the elegance of the prototype $K = 3, 4$, and 7 CP creations.

Conclusions, implications, and future research

Christaller's canonical $K = 3, 4$, and 7 CP systems are mathematically and geographically distinct across multiple dimensions. Geometrically, each K value imposes a unique hierarchical pattern on hexagonal lattices, producing differing SA propagation throughout its network: $K = 3$ emphasizes contiguous market overlap, $K = 4$ aligns interactions along transport axes, and $K = 7$ enforces strict hierarchical containment. Spatial statistical analyses using LISA and GMESF confirm systematic differences in both local and global SA within these separate hypothetical geographic landscapes, whereas adopting a number theory perspective envisions these K values as prime number generators of canonical Löschian numbers. Together, these distinctions provide a powerful multidimensional framework for characterizing and discriminating among CP hierarchies.

These findings underscore that CPT is not purely a descriptive model, but rather an encoding of precise geometric, functional, and algebraic rules governing settlement organization. Recognizing the unique structural signatures of each legitimate K value allows for more informed interpretations of empirical settlement

patterns, spatial interactions, and regional conceptualizations. Furthermore, integrating number-theoretic perspectives illuminates the combinatorial foundations of hierarchical systems, highlighting the relevance of prime-number-based nesting for comprehending both canonical and composite settlement structures.

Finally, building upon this framework, future studies need to investigate non-canonical Löschian numbers, such as 13, 19, or 25, to determine their geographical and hierarchical implications and spatial organization principles. Expanding analyses to larger, irregular, or mixed K value systems could test the generalizability of Christallerian principles beyond idealized lattices. Additionally, applying methods from abstract algebra, lattice theory, number theory, and combinatorics in conjunction with spatial statistical tools may uncover new mechanisms governing settlement hierarchy formation and evolution, bridging mathematical rigor and geographic thought. Finally, these same scenarios need to be evaluated based upon a square, rather than a triangular/hexagonal, lattice, a topic first broached by Lösch (1954, p. 111).

Conflict of interest

The author declares no conflicts of interest and assures that the work is the result of his own creation.

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Appendix A. Central place theory assumptions

1. Space is isotropic. The terrain is flat, without natural barriers, and transport conditions are uniform in all directions.
2. A uniform geographic population distribution. An even population density and comparable purchasing power across space.
3. Consumers behave rationally. Choosing the nearest CP to obtain goods and services to minimize transport costs.
4. Goods and services differ in their market range. Everyday goods, such as bread, have a small spatial range, whereas specialized services, such as higher education or advanced medical care, have a larger range.
5. A threshold level of service profitability exists. Each service requires a minimum number of customers, with more specialized services having larger thresholds.
6. CPs form a hierarchy. Smaller settlements provide basic goods and services, while the largest cities serve the widest areas and supply the most advanced central functions.
7. Service/market areas have a hexagonal shape. A combining of adjacent equilateral triangles.
8. Hierarchical CP systems arise from one of the following three principles:
 - market (minimizing consumer travel distance),
 - transportation (optimizing transport costs), and
 - administrative (lower-order centers are fully contained within higher-order center catchment areas).

Appendix B. Counting hierarchy level central places

Partial theoretical CP map construction begins by defining a two-dimensional square integer lattice via the number of horizontal axis points from its center, the hierarchy-pinnacle CP position. Next, applying the Eisenstein transformation produces the integer mapping of each lattice point (u, v) to a complex number in the hexagonal plane, such that the new point is $(x = u + v/2, y = \sqrt{3}v/2)$, creating the necessary triangular lattice; these are the Tier 1 settlements. Embedding this landscape in a hexagonal-shaped region is convenient. Finally, constructing the individual hierarchical level sublattices – each planar with evenly spaced points (à la the Euclidean metric) in a triangular tiling, ensuring equilateral spacing between nearest neighbors, reflecting optimal coverage – requires solving the quadratic Diophantine equation $K = a^2 - ab + b^2$, $0 < b < a$, and then generating sublattices with modular arithmetic as follows:

even levels: $\text{MOD}(au + bv, K) = 0$ & $\text{MOD}[(a - b)u + av, K] = 0$, and
odd levels: $\text{MOD}(u, K) = 0$ & $\text{MOD}(v, K) = 0$.

A geographic information system (GIS) can map every set of tier points. A Thiesen/Voronoi polygon GIS tool can demarcate market areas.

Niektóre podstawowe różnice między systemami Christallera dla $K = 3, 4$ i 7: autokorelacja przestrzenna w praktyce

Zarys treści: Artykuł podkreśla różnice we właściwościach trzech kanonicznych modeli miejsc centralnych Christallera ($K = 3, 4$ i 7), z uwzględnieniem autokorelacji przestrzennej (SA). Przełomowe spostrzeżenia to: (1) implikacje wynikające z rozsądnie zwiększonej minimalnej wielkości krajobrazu geograficznego, towarzyszącej sześciopoziomowej hierarchii dla trzech wartości K ; (2) SA sąsiedztwa dominuje nad SA hierarchiczną; (3) dominuje ujemna SA; (4) inflacja wariancji SA w liczebności populacji jest znaczna, maskując losowy charakter zmiennej Poissona w systemie populacji miast; (5) korzystne jest badanie krajobrazów geograficznych znacznie większych niż przedstawione w większości publikacji naukowych. Analizy lokalnych wskaźników powiązania przestrzennego (LISA) oraz filtrowania przestrzennego wektorów własnych Morana (MESF) potwierdzają różnice SA w odrębnych idealnych krajobrazach geograficznych.

Słowa kluczowe: teoria centralizacji, Christaller, filtrowanie przestrzenne wektorów własnych Morana, autokorelacja przestrzenna, rozkład Sui-Poissona

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